Problem Set 1: Detailed Answer Key & Derivations

CBE 30235: Introduction to Nuclear Engineering

Problem 1: Nuclear Density of a Pulsar

Goal: Estimate the radius of a neutron star using the liquid-drop model approximation ($R \approx R_0 A^{1/3}$).

Step 1: Determine the total number of nucleons (A) The star is composed of neutrons. We equate the total mass of the star to the mass of the constituent nucleons.

$$M_{\text{star}} = 1.4 \times M_{\odot}$$
$$M_{\text{star}} = 1.4 \times (1.989 \times 10^{30} \text{ kg}) = 2.785 \times 10^{30} \text{ kg}$$

The number of nucleons (A) is the total mass divided by the neutron mass (m_n):

$$A = \frac{M_{\text{star}}}{m_n} = \frac{2.785 \times 10^{30} \text{ kg}}{1.675 \times 10^{-27} \text{ kg}} \approx 1.662 \times 10^{57} \text{ nucleons}$$

Step 2: Apply the Nuclear Radius Formula Assuming the star maintains standard nuclear density (like a giant nucleus):

$$R = R_0 A^{1/3}$$
$$R = (1.25 \times 10^{-15} \text{ m}) \times (1.662 \times 10^{57})^{1/3}$$

Calculate the cube root: $(1.662 \times 10^{57})^{1/3} \approx 1.184 \times 10^{19}$.

$$R = (1.25 \times 10^{-15}) \times (1.184 \times 10^{19})$$
$$R \approx 14,800 \text{ m}$$

Final Answer: The radius is approximately **14.8 km**. *(Discussion: This matches the known size of neutron stars, which are typically 10–15 km in radius, verifying that they are essentially macroscopic atomic nuclei held together by gravity.)*

Problem 2: Mass Defect of Alpha Particle

Goal: Calculate Binding Energy using the neutral atom mass convention.

Step 1: Set up the Mass Defect Equation (Δm) We use neutral atomic masses so that electron masses cancel out automatically.

$$\Delta m = \left[Z \cdot M(^1\text{H}) + N \cdot m_n\right] - M(^A_Z\text{X})$$

For $^4_2$He ($Z = 2, N = 2$):

- $2 \times M(^1\text{H}) = 2 \times 1.007825 \text{ u} = 2.015650 \text{ u}$
- $2 \times m_n = 2 \times 1.008665 \text{ u} = 2.017330 \text{ u}$
- Sum of parts $= 4.032980$ u
- Mass of ^{4}He $= 4.002603$ u

$$\Delta m = 4.032980 - 4.002603 = \mathbf{0.030377} \text{ u}$$

Step 2: Convert to Energy Using the conversion factor $c^2 = 931.5$ MeV/u:

$$BE = \Delta m \times 931.5 = 0.030377 \times 931.5 = \mathbf{28.30} \text{ MeV}$$

Step 3: Binding Energy per Nucleon (E_b)

$$E_b = \frac{BE}{A} = \frac{28.30 \text{ MeV}}{4} = \mathbf{7.07} \text{ MeV/nucleon}$$

Conceptual Answer: We use $M(^1\text{H})$ instead of the proton mass m_p because the standard table lists **neutral atomic masses**.

- Reactants side: $2 \times M(^1\text{H})$ contains 2 protons + 2 electrons.
- Products side: $M(^4\text{He})$ contains 2 protons + 2 neutrons + 2 electrons.
- The electron masses cancel exactly ($2m_e - 2m_e = 0$), allowing us to ignore the electron binding energy (which is negligible, $\sim$eV vs MeV).

Problem 3: Physics of Tc-99m Recoil

Goal: Calculate photon wavelength and nuclear recoil energy.

Part (a): Gamma Wavelength Using the relationship $E = hf = hc/\lambda$:

$$\lambda = \frac{hc}{E} \approx \frac{1240 \text{ eV} \cdot \text{nm}}{140,500 \text{ eV}}$$

$$\lambda \approx 0.008825 \text{ nm} = \mathbf{8.83} \text{ pm}$$

Part (b): Recoil Energy Calculation Momentum must be conserved. The nucleus recoils with momentum equal and opposite to the photon: $p_{nuc} = p_\gamma = E_\gamma/c$. The kinetic energy of the recoiling nucleus is $E_r = \frac{p^2}{2M}$. Substituting $p = E_\gamma/c$:

$$E_r = \frac{(E_\gamma/c)^2}{2M} = \frac{E_\gamma^2}{2Mc^2}$$

- $E_\gamma = 0.1405$ MeV
- Mass of Tc-99 ≈ 99 u.
- Rest Mass Energy $Mc^2 = 99 \times 931.5 \text{ MeV} \approx 92,218$ MeV.

$$E_r = \frac{(0.1405)^2}{2 \times 92218} = \frac{0.01974}{184436} \approx 1.07 \times 10^{-7} \text{ MeV}$$

Convert to eV:

$$E_r = 1.07 \times 10^{-7} \text{ MeV} = \mathbf{0.107} \text{ eV}$$

Part (c): Bond Comparison

- **Recoil Energy:** 0.107 eV.
- **Bond Energy:** Typical covalent bonds are $2.0 - 5.0$ eV.
- **Conclusion:** No, the recoil energy from this specific gamma emission is insufficient to break the chemical bond. The Technetium atom will likely remain attached to the pharmaceutical molecule.

Problem 4: Valley of Stability

Values:

- ^{12}C: $Z = 6, N = 6 \implies N/Z = 1.0$
- ^{56}Fe: $Z = 26, N = 30 \implies N/Z \approx 1.15$
- ^{208}Pb: $Z = 82, N = 126 \implies N/Z \approx 1.54$

Physical Reasoning: Inside the nucleus, there is a competition between the **Strong Nuclear Force** (attractive) and the **Coulomb Force** (repulsive).

1. The Strong Force is very short-range (~ 1 fm), acting only between immediate neighbors. Adding more protons/neutrons essentially adds attraction linearly (saturation).
2. The Coulomb Force is long-range ($\propto 1/r$). Every proton repels every other proton in the nucleus, so repulsion grows as $\sim Z^2$.
3. As Z increases, the repulsion grows faster than the attraction. To compensate, heavy nuclei require an excess of neutrons ($N > Z$) to provide additional "glue" (Strong Force attraction) without adding any further Coulomb repulsion.

Problem 5: Neutron Decay Q-Value

Reaction: $n \to p + e^- + \bar{\nu}_e$

Q-Value Equation:

$$Q = [m_{\text{initial}} - m_{\text{final}}]c^2 = [m_n - (m_p + m_e)]c^2$$

Using the neutral atom trick, we know that $M(^1\text{H}) \approx m_p + m_e$ (neglecting the 13.6 eV electron binding energy, which is far smaller than the nuclear scale). Substituting $M(^1\text{H})$ for $(m_p + m_e)$:

$$\Delta m = m_n - M(^1\text{H})$$

- $m_n = 1.008665$ u
- $M(^1\text{H}) = 1.007825$ u

$$\Delta m = 1.008665 - 1.007825 = 0.000840 \text{ u}$$

$$Q = 0.000840 \text{ u} \times 931.5 \text{ MeV/u} = \mathbf{0.782} \text{ MeV}$$

Since $Q > 0$, the reaction is exothermic and spontaneous. A free neutron will eventually decay (Mean lifetime $\tau \approx 880$ s).